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Normal modes and quality factors of spherical dielectric resonators: I – Shielded dielectric sphere

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Abstract. Electromagnetic theoretic analysis of shielded homogeneous and isotropic dielectric spheres has been made. Characteristic equations for the TE and TM modes have been derived. Dielectric spheres of radii of the order of μm size are found suitable for the optical frequency region whereas for the microwave region radii of the order of mm size are found suitable. Parameters suitable for their application in the optical and microwave frequency ranges have been used to compute the frequencies corresponding to the normal modes for the TE and TM modes. Expressions for the quality factors for realistic resonators, i.e., for a dielectric sphere with a non-zero conductivity and a metal shield with a finite conductivity have also been derived for the TE and TM modes. Computations of the quality factors have been made for resonators with parameters suitable for the optical and the microwave regions.

Keywords. Eigenmodes; spherical resonators; spherical dielectric resonators; quality factors.

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1. Introduction

The earliest reported study on spherical resonators seems to be that of Debye [1] wherein he has studied normal modes of a conducting sphere embedded in a perfect dielectric medium. Stratton has treated the above case and the case of oscillations of a spherical cavity in his classic text [2]. While discussing the cavity resonators, Waldron [3] has considered the case of the spherical homogeneous simple perfect cavity and has derived the field expressions and the characteristic equations for such a cavity based on the treatment of Bromwich [4]. A dielectric sphere with a given dielectric constant and radius possesses natural modes of oscillation having characteristic frequencies. Such oscillations are known as structure resonances and these have been studied both theoretically and experimentally in the microwave region [5,6] and more recently in the optical region of the electromagnetic spectrum [7–10].

Structure resonances have been studied using fluorescence [11], optical levitation [12], absorption [13] and scattering [14].

Electromagnetic field analysis of spherical dielectric resonators has been presented by a number of workers [15]. In all the above cases the resonant frequencies and the quality factors have been computed for the resonators with parameters suitable for the microwave region. From the survey of the published literature it seems that no theoretical and/or experimental studies are available for the optical frequency region. In addition neither the normal mode frequencies nor the quality factors in the optical region seem to have been reported for shielded spherical dielectric resonators. Shielded resonators have drastically reduced quality factors due to metallic loss of the shield and dielectric loss of the dielectric medium. However, cavity resonators with superconducting walls have been found to have a Q factor as high as 10^9 at cryogenic temperatures [16]. In the present work, electromagnetic field analysis for the shielded dielectric resonators has been presented. Expressions for the field components, the characteristic equations and the quality factors have been derived. The resonant frequencies and the quality factors have been computed for the optical and microwave regions.

2. Theory

The shielded homogeneous and isotropic spherical dielectric resonator is in principle equivalent to a spherical hole in a perfect conductor, filled with the dielectric material. Waldron [3] has presented the analysis for a spherical cavity in a perfect conductor. However, the procedure followed by Waldron [3] seems to be rather clumsy. In the present work eigenmodes of a spherical homogeneous and isotropic dielectric resonator enclosed in a metallic spherical shell are determined using straightforward procedure. In a source-free homogeneous and dielectric medium the four Maxwell's equations are given by

$$\vec{\nabla} \cdot \vec{D} = 0, \tag{1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1c}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}. \tag{1d}$$

Constitutive relations for $\vec{B}$ and $\vec{D}$ are given by

$$\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}, \tag{2a}$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}, \tag{2b}$$

where μ_0 and ε_0 are respectively the permeability and permittivity of the free space, μ and ε are the corresponding quantities for the dielectric material and $\mu_r = \mu/\mu_0$, $\varepsilon_r = \varepsilon/\varepsilon_0$. For a non-magnetic dielectric, $\mu_r = 1$ and hence, $\vec{B} = \mu_0 \vec{H}$.

Assuming $e^{j\omega t}$ time dependence for $\vec{E}$ and $\vec{H}$ and using eqs (2a) and (2b), eqs (1a)–(1d) reduce to

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{3a}$$

$$\vec{\nabla} \cdot \vec{H} = 0, \tag{3b}$$

$$\vec{\nabla} \times \vec{E} = -j\omega\mu_0 \vec{H}, \tag{3c}$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon_r\varepsilon_0 \vec{E}. \tag{3d}$$

Using expression for the curl in the spherical polar coordinates system (Appendix A, eq. (A1)) eqs (3c) and (3d) give

$$j\omega\mu_0 H_r = -\frac{1}{r^2 \sin\theta}\left\{\frac{\partial(r\sin\theta E_\phi)}{\partial\theta} - \frac{\partial(rE_\theta)}{\partial\phi}\right\}, \tag{4a}$$

$$j\omega\mu_0 H_\theta = -\frac{1}{r \sin\theta}\left\{\frac{\partial(E_r)}{\partial\phi} - \frac{\partial(r\sin\theta E_\phi)}{\partial r}\right\}, \tag{4b}$$

$$j\omega\mu_0 H_\phi = -\frac{1}{r}\left\{\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial(E_r)}{\partial\theta}\right\}, \tag{4c}$$

$$j\omega\varepsilon_r\varepsilon_0 E_r = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial(r\sin\theta H_\phi)}{\partial\theta} - \frac{\partial(rH_\theta)}{\partial\phi}\right\}, \tag{5a}$$

$$j\omega\varepsilon_r\varepsilon_0 E_\theta = \frac{1}{r \sin\theta}\left\{\frac{\partial(H_r)}{\partial\phi} - \frac{\partial(r\sin\theta H_\phi)}{\partial r}\right\}, \tag{5b}$$

$$j\omega\varepsilon_r\varepsilon_0 E_\phi = \frac{1}{r}\left\{\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial(H_r)}{\partial\theta}\right\}. \tag{5c}$$

Equations (4a)–(4c) can be used to find magnetic field components provided the field electric components are known. Similarly, eqs (5a)–(5c) can be used to find field electric components provided the magnetic field components are known. Now taking the curl of eq. (3c) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega\mu_0(\vec{\nabla} \times \vec{H}). \tag{6}$$

Using the vector identity (eq. (A2)), and using eqs (3a) and (3d) we get

$$\left(\nabla^2 + \mu_0\varepsilon_0\varepsilon_r\omega^2\right)\vec{E} = 0. \tag{7}$$

Similarly, taking the curl of eq. (3d) and using eqs (3b) and (3c) we get

$$\left(\nabla^2 + \mu_0\varepsilon_0\varepsilon_r\omega^2\right)\vec{H} = 0. \tag{8}$$

Equations (7) and (8) represent differential equations for the electric vector $\vec{E}$ and the magnetic vector $\vec{H}$ respectively. In the following we use the standard theory

[17–20] to find the electric and magnetic fields for the TE and TM modes separately. For the TE mode the following condition is satisfied:

$$\vec{r} \cdot \vec{E} = 0. \tag{9}$$

The electric field ($\vec{E}$) can be written in terms of the gradient of some scalar function ψ as

$$\vec{E} = \vec{r} \times \vec{\nabla}\psi. \tag{10}$$

Evidently, eq. (10) satisfies condition (9) (one may verify this using eqs (9) and (10) and the vector identity (A3)). Here, ψ is any well-behaved scalar field that satisfies the Helmholtz equation (Appendix B, eq. (B1)). Using ψ determined in the Appendix B (eq. (B11)) components of the electric field can be determined using eq. (10) and the expression of $\vec{\nabla}\psi$ given in eq. (A4) as

$$E_r = 0, \tag{11a}$$

$$E_\theta = \frac{mA}{\sqrt{r}\sin\theta} J_{n+(1/2)}(kr) P_n^m(\cos\theta)\sin m\phi, \tag{11b}$$

$$E_\phi = \frac{A}{\sqrt{r}} J_{n+(1/2)}(kr) \frac{\mathrm{d}}{\mathrm{d}\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi. \tag{11c}$$

Alternatively, introducing the angular momentum operator $\vec{L}$ defined as, $\vec{L} = (1/j)(\vec{r} \times \vec{\nabla})$, where $j = \sqrt{-1}$, and constructing L^2 and its relationship with the Laplacian operator (∇^2), the solution for TE mode can be constructed following Jackson [20]. Both the methods yield equivalent results as can be verified from the field expressions (11a)–(11c) and the ones given by Jackson [20].

Now substituting the values of E_r, E_θ and E_ϕ from eqs (11a)–(11c) into eqs (4a)–(4c), one obtains expressions for H_r, H_θ and H_ϕ. The RHS of eq. (4a) involves E_θ and E_ϕ and substituting the values of E_θ and E_ϕ from eqs (11b) and (11c) it yields

$$H_r = -\frac{AJ_{n+(1/2)}(kr)\cos m\phi}{j\omega\mu_0 r^{3/2}} \times \left[\begin{array}{r} \sin^2\theta \dfrac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left\{P_n^m(\cos\theta)\right\} - 2\cos\theta\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left\{P_n^m(\cos\theta)\right\} \\ -\dfrac{m^2}{\sin^2\theta}\left\{P_n^m(\cos\theta)\right\} \end{array}\right]. \tag{12}$$

Using recurrence relations for $P_n^m(\cos\theta)$ (Appendix C, eqs (C1) and (C2)) the term within the square bracket of eq. (12) is simplified to give $-n(n+1)P_n^m(\cos\theta)$. Therefore, the expression for H_r becomes

$$H_r = \frac{n(n+1)A}{j\omega\mu_0 r^{3/2}} J_{n+(1/2)}(kr) P_n^m(\cos\theta)\, \cos\, m\phi. \tag{12a}$$

To get the expressions for E_θ and E_ϕ is straightforward, as RHSs of eqs (4b) and (4c) involve E_r which vanishes for the TE mode leaving single term for these equations. The expressions for E_θ and E_ϕ are determined as

$$H_\theta = \frac{A}{j\omega\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{n+(1/2)}(kr)\right\}\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi, \tag{12b}$$

$$H_\phi = -\frac{mA}{j\omega\mu_0 r\sin\theta}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{n+(1/2)}(kr)\right\}P_n^m(\cos\theta)\sin m\phi. \tag{12c}$$

Similarly, for the TM modes the field components are given by

$$E_r = \frac{n(n+1)A}{j\omega\varepsilon_0\varepsilon_r r^{3/2}}J_{n+(1/2)}(kr)P_n^m(\cos\theta)\cos m\phi, \tag{13a}$$

$$E_\theta = \frac{A}{j\omega\varepsilon_0\varepsilon_r r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{n+(1/2)}(kr)\right\}\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi, \tag{13b}$$

$$E_\phi = -\frac{mA}{j\omega\varepsilon_0\varepsilon_r r\sin\theta}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{n+(1/2)}(kr)\right\}P_n^m(\cos\theta)\sin m\phi, \tag{13c}$$

$$H_r = 0, \tag{14a}$$

$$H_\theta = -\frac{mA}{\sqrt{r}\sin\theta}J_{n+(1/2)}(kr)P_n^m(\cos\theta)\sin\, m\phi, \tag{14b}$$

$$H_\phi = -\frac{A}{\sqrt{r}}J_{n+(1/2)}(kr)\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{P_n^m(\cos\theta)\right\}\cos m\phi. \tag{14c}$$

It must be mentioned here that the field expressions obtained in the present case differ from those obtained by Waldron [3] by a factor of $j\omega\mu_0$ for the TE modes and by a factor of $j\omega\varepsilon_0\varepsilon_r$ for the TM modes.

3. Characteristic equations for the TM and the TE modes

Having determined the electric and the magnetic field expressions for the TE and TM modes, one can derive the characteristic or the eigenvalue equations for these modes employing the boundary conditions. On the metal surface, i.e., $r = a$, the electric and the magnetic fields satisfy the boundary conditions, that the tangential components of $\vec{E}$ are zero for all θ and ϕ, hence, for the TM modes one gets from eqs (13b) and (13c),

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{n+(1/2)}(kr)\right\}\right]_{r=a} = 0. \tag{15}$$

On differentiation, eq. (15) yields

$$J_{n+(1/2)}(kr) + 2krJ'_{n+(1/2)}(kr) = 0, \tag{16}$$

where the dash on J denotes derivative with respect to the argument.

Similarly, for the TE modes one has from eqs (11b) and (11c)

$$J_{n+(1/2)}(kr)\big|_{r=a} = 0. \tag{17}$$

Equations (16) and (17) are the eigenvalue or the characteristic equations for the TM and TE modes respectively. Using the relations (D1) and (D2), eqs (16) and (17) are simplified to become respectively,

$$ka\,\{j_{n+1}(ka) - j_{n-1}(ka)\} - j_n(ka) = 0, \tag{18}$$

$$j_n(ka) = 0. \tag{19}$$

It should be noted here that though the field expressions obtained in the present case (eqs (11a)–(11c), (12a)–(12c), (13a)–(13c) and (14a)–(14c)) differ from those obtained by Waldron [3], the characteristic equations (eqs (16) and (17) or (18) and (19)) are exactly the same in both the cases. Hence, the resonant frequencies obtained in the present study are essentially the same as obtained by Waldron [3] for the identical input parameters.

4. Stored energy, losses, and quality factor

So far the treatment is based on the assumption that the dielectric material of the sphere is lossless and the shield is perfectly conducting. Under such an idealized condition the quality factor of the resonator is infinite. However, in actual practice neither the dielectric is a lossless material nor the shield is a perfect conductor and hence, the resonator has some finite value of the quality factor. Loss of the stored energy is due to its dissipation through (i) the volume of the dielectric and (ii) the surface of the shield.

4.1 *Stored energy*

The energy W stored in a shielded dielectric medium of permittivity ε and permeability μ is given by

$$W = \varepsilon \iiint\limits_V \vec{E} \cdot \vec{E}^* \,\mathrm{d}V = \mu \iiint\limits_V \vec{H} \cdot \vec{H}^* \,\mathrm{d}V, \tag{20}$$

where V is the volume of the dielectric. For the spherical geometry the volume element is $\mathrm{d}V = r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi$. Hence, the energy stored in the dielectric sphere for the TM and the TE modes can be determined using the field expressions.

For the TE modes, $E_r = 0$ and for $n = 1$ and $m = 0$ one has $P_1^0(\cos\theta) = \cos\theta$ (eq. (C4)) and hence

$$\left.\begin{aligned} &E_\theta = 0 \\ &E_\phi = -\frac{1}{\sqrt{r}} A J_{3/2}\,(kr) \sin\theta \\ &E_\phi \cdot E_\phi^* = \frac{A^2 J_{3/2}^2(kr) \sin^2\theta}{r} \end{aligned}\right\}. \tag{21}$$

Hence, the energy stored in the resonator for the TE_{10l} mode is given by

$$W_1 = \varepsilon_0 \varepsilon_r A^2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin^3\theta \,\mathrm{d}\theta \int_0^a r J_{3/2}^2(kr)\mathrm{d}r. \tag{22}$$

Using the value of the integral involving r from eq. (D3) and using the relations (D1), (D4)–(D6) eq. (22) on simplification becomes

$$W_1 = \frac{8}{3}\varepsilon_0 \varepsilon_r k A^2 a^3 \left\{ j_1^2(ka) - j_0(ka) j_2(ka) \right\}. \tag{23a}$$

Similarly, for the TE_{20l} and TE_{30l} modes, the expressions for the energy stored in the dielectric sphere are given by

$$W_2 = \frac{24}{5}\varepsilon_0 \varepsilon_r k A^2 a^3 \left\{ j_2^2(ka) - j_1(ka) j_3(ka) \right\}, \tag{23b}$$

$$W_3 = \frac{48}{7}\varepsilon_0 \varepsilon_r k A^2 a^3 \left\{ j_3^2(ka) - j_2(ka) j_4(ka) \right\}. \tag{23c}$$

Expressions for the energy W' stored in the resonator for the TM modes are determined using the following equation:

$$W' = \mu_0 \int_0^{2\pi} \int_0^{\pi} \int_0^a (\vec{H} \cdot \vec{H}^*)\, r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi. \tag{24}$$

For the TM_{10l} modes $n = 1$, $m = 0$ and hence,

$$\left.\begin{aligned} H_r &= 0 \\ H_\theta &= 0 \\ H_\phi &= \frac{A}{\sqrt{r}} J_{3/2}(kr)\sin\theta \\ H_\phi \cdot H_\phi^* &= \frac{A^2}{r} J_{3/2}^2(kr)\, \sin^2\theta \end{aligned}\right\}. \tag{25}$$

Equations (24) and (25) give us

$$W_1' = \mu_0 A^2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin^3\theta \,\mathrm{d}\theta \int_0^a r J_{3/2}^2(kr)\mathrm{d}r. \tag{26}$$

Equation (26) on simplification gives us

$$W_1' = \frac{8}{3}\mu_0 k A^2 a^3 \left\{ j_1^2(ka) - j_0(ka) j_2(ka) \right\}. \tag{27a}$$

Similarly, for the TM_{20l} and TM_{30l} modes, the energy expressions are given by

$$W_2' = \frac{24}{5}\mu_0 k A^2 a^3 \left\{ j_2^2(ka) - j_1(ka) j_3(ka) \right\}, \tag{27b}$$

$$W_3' = \frac{48}{7}\mu_0 k A^2 a^3 \left\{ j_3^2(ka) - j_2(ka) j_4(ka) \right\}. \tag{27c}$$

4.2 *Losses*

The power loss P of the system is defined as

$$P = P_{\mathrm{m}} + P_{\mathrm{d}}, \tag{28}$$

where P_{m} is the metallic loss due to a finite high conductivity of the metallic shield and P_{d} is the dielectric loss due to its finite small conductivity.

The metallic loss P_{m} is a surface phenomenon and it occurs due to finite conductivity of the metal shield. Let R_{s} be the surface resistivity defined by

$$R_{\mathrm{s}} = \frac{1}{\sigma\, d}, \tag{29}$$

where σ = metallic conductivity and d = skin depth given by

$$d = \sqrt{2/\omega\mu_0\sigma}. \tag{30}$$

The metallic loss P_{m} depends on R_{s} and is given by

$$P_{\mathrm{m}} = R_{\mathrm{s}} \iint\limits_{\text{metallic surface}} (\vec{H}\cdot\vec{H}^{*}\,\mathrm{d}S)_{r=a}, \tag{31}$$

where $\mathrm{d}S = r^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$ is the surface area element.

The dielectric loss is a volume phenomenon and is a result of finite non-zero (though small) conductivity (σ_{d}) of the dielectric material which is given by

$$\sigma_{\mathrm{d}} = \omega\varepsilon \tan\delta, \tag{32}$$

where $\tan\delta$ is known as the loss tangent of the material and is given by

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'}. \tag{33}$$

Here ε' and ε'' are the real and the imaginary parts of the permittivity ε, i.e.,

$$\varepsilon = \varepsilon' + j\varepsilon''. \tag{34}$$

Thus, a finite non-zero conductivity of the dielectric medium results in a complex permittivity of the medium. The loss tangent is also defined as

$$\tan\delta = \frac{\sigma_{\mathrm{d}}}{\omega\varepsilon} \tag{35}$$

with the complex permittivity ε given by

$$\varepsilon = \varepsilon_0 \left(1 - \frac{j\sigma_{\mathrm{d}}}{\omega\varepsilon_0}\right). \tag{36}$$

From eqs (34) and (36) $\varepsilon' = \varepsilon_0$ and $\varepsilon'' = \sigma_{\mathrm{d}}/\omega$. In terms of σ_{d} the energy lost in the dielectric medium P_{d} is given by

$$P_{\mathrm{d}} = \sigma_{\mathrm{d}} \iiint_V \vec{E} \cdot \vec{E}^* \, \mathrm{d}V. \tag{37}$$

The expression for the energy loss on the metal surface of the sphere can be calculated as follows. For the TE modes, for $n = 1$ and $m = 0$ one has, $H_r = 0$ since, solution of eq. (17) demands that $J_{3/2}(ka) = 0$. Moreover, $H_\phi = 0$ and H_θ is given by

$$H_\theta = -\frac{A}{j\omega\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\sqrt{r}J_{3/2}\left(kr\right)\right\}\sin\theta. \tag{38}$$

From eq. (38) one has

$$H_\theta H_\theta^* = \frac{1}{4r^3\omega^2\mu_0^2}A^2\{J_{3/2}(kr) + 2kr\,J'_{3/2}(kr)\}^2\sin^2\theta. \tag{39}$$

Equations (31) and (39) give expression of energy lost for the TE_{10l} mode as

$$P_{\mathrm{m},1} = \sqrt{\frac{2\omega\mu_0}{\sigma}}\int_0^{2\pi}\int_0^{\pi}\frac{A^2}{4r^3\omega^2\mu_0^2}\{J_{3/2}(kr) + 2kr\,J'_{3/2}(kr)\}^2 \times r^2\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{40}$$

Now, since at $r = a$, $J_{3/2}(ka) = 0 \Rightarrow j_1(ka) = 0$, using relation (D7) and evaluating the integrals, eq. (40) yields

$$P_{\mathrm{m},1} = \frac{16}{3}\sqrt{\frac{1}{2\sigma\omega^3\mu_0^3}}\,A^2a^2k^3j_1'^2(ka). \tag{41a}$$

Similarly, for the TE_{20l} and TE_{30l} modes, the expressions for the energy loss on the metal surface of the sphere are given by

$$P_{\mathrm{m},2} = \frac{48}{5}\sqrt{\frac{1}{2\sigma\omega^3\mu_0^3}}\,A^2a^2k^3j_2'^2(ka), \tag{41b}$$

$$P_{\mathrm{m},3} = \frac{96}{7}\sqrt{\frac{1}{2\sigma\omega^3\mu_0^3}}\,A^2a^2k^3j_3'^2(ka). \tag{41c}$$

The expressions for the dielectric loss for the TE_{10l}, TE_{20l} and TE_{30l} modes can be calculated using eqs (20), (23), (35), and (37) as

$$P_{\mathrm{d},1} = \omega W_1 \tan\delta, \tag{42a}$$

$$P_{\mathrm{d},2} = \omega W_2 \tan\delta, \tag{42b}$$

$$P_{d,3} = \omega W_3 \tan\delta. \tag{42c}$$

Similarly, the expressions for the energy loss on the metal surface for the TM modes are given by

$$P'_{\mathrm{m},1} = \frac{16}{3}\sqrt{\frac{\omega\mu_0}{2\sigma}}A^2a^2k\, j_1^2(ka), \tag{43a}$$

$$P'_{\mathrm{m},2} = \frac{48}{5}\sqrt{\frac{\omega\mu_0}{2\sigma}}A^2a^2k\, j_2^2(ka), \tag{43b}$$

$$P'_{\mathrm{m},3} = \frac{96}{7}\sqrt{\frac{\omega\mu_0}{2\sigma}}A^2a^2k\, j_3^2(ka). \tag{43c}$$

Finally the expressions for the dielectric loss for the TE_{10l}, TE_{20l} and TE_{30l} modes can be calculated from eqs (20), (27), (35) and (37) as

$$P'_{\mathrm{d},1} = \omega\, W'_1 \tan\delta, \tag{44a}$$

$$P'_{\mathrm{d},2} = \omega\, W'_2 \tan\delta, \tag{44b}$$

$$P'_{\mathrm{d},3} = \omega\, W'_3 \tan\delta. \tag{44c}$$

4.3 *Quality factor*

The quality factor Q is defined as

$$Q = \frac{\text{Energy stored}}{\text{Energy loss per cycle}}, \tag{45}$$

$$Q = \omega\frac{W}{P_\mathrm{m} + P_\mathrm{d}}, \tag{46}$$

where W is the energy stored in dielectric medium, P_m is the metallic loss and P_d is the dielectric loss. With the help of eq. (46) one can calculate the quality factor for the TE and TM modes. The expression of the quality factor Q_n for the TE modes is obtained using eqs (23), (41), (42) and (46) as

$$Q_n = \frac{1}{\tan\delta + g f_n(j)}, \tag{47}$$

where $n = 1, 2$ and 3 for the TE_{10l}, TE_{20l} and TE_{30l} modes respectively and g and $f_n(j)$ are given by

$$g = \sqrt{\frac{2\sqrt{\varepsilon_r}}{\sigma\mu_0 a c x_{nl}}}, \tag{48}$$

$$f_n(j) = \frac{j'^2_n(ka)}{j_n^2(ka) - j_{n-1}(ka)j_{n+1}(ka)}. \tag{49}$$

Here, x_{nl} is the lth root of eqs (18) and (19) for the TM and TE modes respectively for a given value of n and the dash on j_n in eq. (49) denotes derivative with respect to its argument. Similarly, the expression of the quality factor Q_n' for the TM modes is determined using eqs (27), (43), (44) and (46) and it is given by

$$Q_n' = \frac{1}{\tan\delta + g f_n'(j)}, \tag{50}$$

where $f_n'(j)$ is given by

$$f_n'(j) = \frac{j_n^2(ka)}{j_n^2(ka) - j_{n-1}(ka) j_{n+1}(ka)}. \tag{51}$$

It must be noted here that although the expressions (47) and (50) have been derived on the basis of the values of $n = 1, 2$ and 3, these are valid for any value of n.

5. Numerical calculations

Solving eqs (18) and (19) gives the value of ka as

$$ka = \frac{2\pi a}{\lambda} = \frac{2\pi a}{\lambda_0}\sqrt{\varepsilon_r}, \tag{52}$$

where λ_0 is the value of free-space wavelength and λ is the wavelength in dielectric at which the resonator oscillates in the appropriate mode. The frequency of oscillation is given by

$$\nu_{nl} = \frac{x_{nl} c}{2\pi a\sqrt{\varepsilon_r}}, \tag{53}$$

where x_{nl} is the value of $2\pi a/\lambda = ka$ which satisfies eq. (18) or (19). If x_{nl} is the lth root of the appropriate equation with a given value of n, the mode is E_{nml} (TMnml) or H_{nml} (TEnml) mode (n is an integer not equal to zero, $m = 0, 1, 2, \ldots, n$). It must be noted that the eigenvalue equation and so the frequency does not depend on m, i.e., ϕ. Therefore, modes with a given value of n are $n+1$ fold degenerate, i.e., all the modes with the same value of n but with different values of m ($m = 0, 1, \ldots, n$) have the same frequency. However, with a given value of n, eqs (18) and (19) have a number of roots ($l = 1, 2, 3, \ldots$ correspond to the first, second, third, ... roots respectively. Hence, the roots of eqs (18) and (19) and so the frequencies for the TM and TE modes are specified with two indices (n and l) only. In order to solve eqs (18) and (19) one has the value of $j_0(x) = \sin x/x$ and that for the higher values of n $(= 1, 2, 3, \ldots)$ can be calculated from the relation

$$j_{n+1}(x) = -x^n \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{j_n(x)}{x^n}\right\}. \tag{54}$$

Hence, eqs (18) and (19) are simple algebraic equations and can be solved numerically. The first six roots of eqs (18) and (19) are given in tables 1 and 2 respectively. With $a = 1$ μm and $\varepsilon_r = 3.78$ (quartz) the frequencies have been calculated and these are collected in tables 3 and 4 respectively for the TM and TE modes for $n = 1$–6 and $l = 1$–6.

Table 1. Roots of the characteristic equation (eq. (18)) for the TM modes.

	l					
n	1	2	3	4	5	6
1	2.7437	6.1168	9.3166	12.4859	15.6439	18.7963
2	3.8702	7.7431	10.7130	13.9205	17.1027	20.2720
3	4.9734	8.7218	12.0636	15.3136	18.5242	21.7139
4	6.0619	9.9675	13.3801	16.6742	19.9154	23.1278
5	7.1402	11.1890	14.6701	18.0085	21.2815	24.5178
6	8.2108	12.3915	15.9387	19.3212	22.6263	25.8873

Table 2. Roots of the characteristic equation (eq. (19)) for the TE modes.

	l					
n	1	2	3	4	5	6
1	4.4934	7.7253	10.9041	14.0662	17.2208	20.3713
2	5.7635	9.0950	12.3229	15.5146	18.6890	21.8539
3	6.9879	10.4171	13.6980	16.9236	20.1218	23.3042
4	8.1826	11.7049	15.0397	18.3013	21.5254	24.7276
5	9.3558	12.9665	16.3547	19.6532	22.9046	26.1278
6	10.5128	14.2074	17.6480	20.9835	24.2628	27.5079

Table 3. Resonant frequencies ($\times 10^{-14}$ Hz) for the TM modes.

	l					
n	1	2	3	4	5	6
1	0.67	1.50	2.29	3.07	3.84	4.62
2	0.95	1.83	2.63	3.42	4.20	4.98
3	1.22	2.14	2.96	3.76	4.55	5.33
4	1.49	2.45	3.29	4.09	4.89	5.68
5	1.75	2.75	3.60	4.42	5.23	6.02
6	2.02	3.04	3.91	4.75	5.56	6.36

Using eqs (50) and (53) the Q values for the TE and TM modes have been calculated for $n = 1$–6 and $l = 1$–6. In order to calculate the quality factors the shield has been assumed to be made up of Cu which has $\sigma = 0.58 \times 10^8/\Omega$m. The tangent loss for the quartz has value 10^{-4} for the optical region. The Q values are collected in tables 5 and 6 for the TM and TE modes respectively. We have also calculated the resonant frequencies and the quality factors for the microwave region for a material [15] with $\varepsilon_r = 9.7$. For this material the value of $\tan\delta = \nu/40000$ (where ν is the frequency in GHz) has been assumed by earlier workers [15]. However, we have taken the value of $\tan\delta$ as 10^{-4} for the microwave region also. The frequencies lie in the range 40–400 GHz for the TM modes and in

Table 4. Resonant frequencies ($\times 10^{-14}$ Hz) for the TE modes.

	l					
n	1	2	3	4	5	6
1	1.10	1.89	2.68	3.45	4.23	5.00
2	1.42	2.23	3.03	3.82	4.59	5.38
3	1.72	2.56	3.36	4.16	4.94	5.72
4	2.01	2.87	3.69	4.49	5.29	6.07
5	2.30	3.18	4.02	4.83	5.62	6.42
6	2.58	3.49	4.33	5.15	5.96	6.76

Table 5. Quality factors for the TM modes.

	l					
n	1	2	3	4	5	6
1	90	173	219	255	286	313
2	88	179	227	264	295	322
3	85	183	233	271	302	329
4	83	186	238	276	308	336
5	82	187	241	281	313	341
6	80	188	244	284	317	345

Table 6. Quality factors for the TE modes.

	l					
n	1	2	3	4	5	6
1	157	204	242	274	302	327
2	177	221	257	290	314	339
3	194	236	270	299	325	349
4	210	250	283	311	336	360
5	224	263	294	322	347	369
6	237	275	305	332	356	378

the range 70–425 GHz for the TE modes. The Q values lie in the ranges 1850–4720 and 4710–4960 for the TM and TE modes respectively. The frequencies and the Q values calculated in the present case for the microwave region agree with those reported in [15].

Appendix A: Some vector relations

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{bmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{bmatrix}, \tag{A1}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2(\vec{A}), \tag{A2}$$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}), \tag{A3}$$

$$\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\ , \tag{A4}$$

$$\vec{\nabla} \cdot \vec{\nabla}\psi = \nabla^2\psi = \frac{1}{r^2\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right\}. \tag{A5}$$

Appendix B: Helmholtz equation and its solution

The Helmholtz equation is given by

$$\left(\nabla^2 + k^2\right)\psi = 0, \tag{B1}$$

where $\psi \equiv \psi\,(r, \theta, \phi)$ is some scalar function.

Using the expression of ∇^2 in the spherical polar coordinate system (eq. (A5)), assuming $\psi\,(r, \theta, \phi) = R(r)\Theta\,(\theta)\,\Phi\,(\phi)$, and employing the method of separation of variables, eq. (B1) gives

$$r^2\sin^2\theta\,\frac{\ddot{R}}{R} + 2r\sin^2\,\theta\,\frac{\dot{R}}{R} + \sin^2\,\theta\frac{\ddot{\Theta}}{\Theta} + \sin\,\theta\,\cos\,\theta\frac{\dot{\Theta}}{\Theta} + \frac{\ddot{\Phi}}{\Phi} + \omega^2\varepsilon_r\varepsilon_0\mu_0 r^2\,\sin^2\,\theta = 0, \tag{B2}$$

where the dots denote derivatives with respect to r, θ, ϕ as the case may be. Since ϕ occurs in $\ddot{\Phi}/\Phi$ only, it can be replaced by a constant $-m^2$, i.e.,

$$\frac{\ddot{\Phi}}{\Phi} + m^2 = 0. \tag{B3}$$

The solutions of eq. (B3) are of the form

$$\Phi(\phi) \sim \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\phi. \tag{B4}$$

Now eq. (B2) is reduced to the following form:

$$\frac{r^2}{R}\left\{\ddot{R} + \frac{2\dot{R}}{r} + \omega^2\varepsilon_r\varepsilon_0\mu_0 R\right\} + \frac{1}{\Theta}\left\{\ddot{\Theta} + \cot\,\theta\,\dot{\Theta} - \frac{m^2\Theta}{\sin^2\theta}\right\} = 0. \tag{B5}$$

Since the first bracketed term contains r only and the second bracketed term contains θ only, these two terms can separately be equated to a constant. Let the first bracketed term be replaced by a constant $n(n+1)$ then one has

$$\ddot{\Theta} + \cot\theta\dot{\Theta} + \left\{ n(n+1) - \frac{m^2}{\sin^2\theta} \right\} \Theta = 0, \tag{B6}$$

$$\ddot{R} + \frac{2\dot{R}}{r} + \left\{ \omega^2 \varepsilon_r \varepsilon_0 \mu_0 - \frac{n(n+1)}{r^2} \right\} R = 0. \tag{B7}$$

The solutions of eq. (B6) are associated Legendré polynomials $P_n^m(\cos\theta)$ and Q_n^m $(\cos\theta)$. However, $Q_n^m(\cos\theta)$ has singularities at $\cos\theta = \pm 1$, leading to infinite values of ψ and hence, of the electromagnetic field. Therefore, the acceptable solution of eq. (B6) is given by

$$\Theta(\theta) \sim P_n^m(\cos\theta). \tag{B8}$$

Substituting $X = R\sqrt{r}$ in eq. (B7), it reduces to

$$\ddot{X} + \frac{\dot{X}}{r} + \left\{ \omega^2 \varepsilon_r \varepsilon_0 \mu_0 - \frac{(n + (1/2))^2}{r^2} \right\} X = 0, \tag{B9}$$

where the dots denote derivatives with respect to r. Equation (B9) is Bessel's differential equation of order $(n + (1/2))$. On changing the variable from r to $(\sqrt{\omega^2 \varepsilon_0 \mu_0 \varepsilon_r})r = (\omega\sqrt{\varepsilon_0 \mu_0}\varepsilon_r)r = \left(\frac{\omega}{c}\sqrt{\varepsilon_r}\right) r = kr$, where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum, eq. (B9) has the solutions of the form $J_{n+(1/2)}(kr)$ or $Y_{n+(1/2)}(kr)$ or a linear combination of these two. Since $Y_{n+(1/2)}(kr)$ has infinite value at $r = 0$, any linear combination of $J_{n+(1/2)}(kr)$ and $Y_{n+(1/2)}(kr)$ also has infinite value at $r = 0$. Hence, the acceptable solution of eq. (B9) is of the form

$$X(r) \sim J_{n+(1/2)}(kr). \tag{B10}$$

Thus, $R(r) \sim J_{n+(1/2)}(kr)/\sqrt{r}$ and Ψ is given by

$$\Psi(r,\theta,\phi) = \frac{A}{\sqrt{r}} J_{n+(1/2)}(kr) P_n^m(\cos\theta) \cos m\phi, \tag{B11}$$

where A is a constant. Here, we have dropped the sin $m\phi$ term without any loss of generality. Since Ψ is a single valued function, it must take the same value for ϕ and $\phi + 2\pi$, so cos $m\phi = \cos m(\phi + 2\pi) = \cos(m\phi + 2m\pi) = \cos m\phi$ only when $m = 0, \pm 1, \pm 2, \pm 3, \ldots$, i.e., m takes integral values and this limits n also to integral values. Negative values of m give the same field distributions as positive and hence, do not provide separate solutions. Similarly, negative values of n also do not produce extra solutions. The constant m must be less than or equal to n. The possibilities that the constant n can or cannot take value as 0 is discussed in the following. However, m can always take 0 value and it gives rotational symmetry about the diameter of a sphere joining the points $\theta = 0°$ and $\theta = 180°$.

Appendix C: Recurrence relations and expressions for some associated Legendré polynomials

$$(1 - x^2)^{1/2} P_n'^m(x) = \frac{1}{2} P_n^{m+1}(x) - \frac{1}{2}(n+m)(n-m+1) P_n^{m-1}(x) \tag{C1}$$

$$P_n^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}} P_n^m(x) + \{n(n+1) - m(m-1)\} P_n^{m-1}(x) = 0, \tag{C2}$$

$$P_n^0(x) = P_n(x), \tag{C3}$$

$$P_1^0(x) = P_1(x) = x = \cos\theta, \tag{C4}$$

$$P_2^0(x) = P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1), \tag{C5}$$

$$P_3^0(x) = P_3(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta). \tag{C6}$$

The dash on P in eq. (C1) denotes its derivative with respect to the argument x.

Appendix D: Bessel functions, spherical Bessel functions and related expressions

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x), \tag{D1}$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+(1/2)}(x), \tag{D2}$$

$$\int_0^a J_\nu^2(kr) r\,\mathrm{d}r = \frac{a^2}{2}\left\{J'^2_\nu(ka) + \left(1 - \frac{\nu^2}{k^2a^2}\right) J_\nu^2(ka)\right\}, \tag{D3}$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = (2\nu/x) J_\nu(x), \tag{D4}$$

$$J'_\nu(x) = (\nu/x) J_\nu(x) - J_{\nu+1}(x), \tag{D5}$$

$$J'_\nu(x) = J_{\nu-1}(x) - (\nu/x) J_\nu(x), \tag{D6}$$

$$J_{n+(1/2)}(x) + 2xJ'_{n+(1/2)}(x) = \sqrt{\frac{8x}{\pi}}\left[j_n(x) + xj'_n(x)\right], \tag{D7}$$

$$j_n(x) + xj'_n(x) = xj_{n-1}(x) - nj_n(x), \tag{D8}$$

$$j'_n(x) = j_{n-1}(x) - \frac{(n+1)}{x} j_n(x). \tag{D9}$$

Here, J and j appearing in eqs (D1)–(D9) are the Bessel and the spherical Bessel functions of the first kind; ν is any integral or non-integral number, n is an integer and the dash on J or j denotes derivative with respect to the argument.

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